Now we note that

$$\begin{aligned} a+b+c &= 2s \\ ab+bc+ca &= s^2+4Rr+r^2 \\ abc &= 4sRr \\ a^2+b^2+c^2 &= 2(s^2-4Rr-r^2). \end{aligned}$$

Therefore the inequality becomes

$$\begin{aligned} \frac{12sRr+4s(s^2-4Rr-r^2)}{2s(s^2+4Rr+r^2)-4sRr}+\frac{1}{2} &\le \frac{R}{r}, \\ \frac{6Rr+2(s^2-4Rr-r^2)}{(s^2+4Rr+r^2)-2Rr}+\frac{1}{2} &\le \frac{R}{r}, \\ \frac{2s^2-2Rr-2r^2}{s^2+2Rr+r^2}+\frac{1}{2} &\le \frac{R}{r}, \\ 2s^2(R-2r)+r(4R^2+4Rr+3r^2-s^2) &\ge 0, \end{aligned}$$

which is true because by Euler's inequality $R \ge 2r$, and by Gerretsen's inequality $s^2 \le 4R^2+4Rr+3r^2$. So the proof is completed.

4597. *Proposed by George Apostolopoulos.*

Let a, b, c be positive real numbers with $a+b+c=1$. Prove that

$$a^2+b^2+c^2+\frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} \ge 2(ab+bc+ca).$$

We received 29 submissions, of which 27 were correct and complete. We present two solutions.

Solution 1, by Arkady Alt.

Let $p=ab+bc+ca$ and $q=abc$, so $a^2+b^2+c^2=1-2p$ and $\dfrac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}=\dfrac{3q}{p}$.

The problem becomes to prove $1-4p+\dfrac{3q}{p} \ge 0$.

Since $3\left(ab+bc+ca\right) \le \left(a+b+c\right)^2$ then $0<p\le\dfrac{1}{3}$, and there are two cases.

If $p \in (0, 1/4]$ since $p, q > 0$ then $1-4p+\dfrac{3q}{p} > 0$.

By Schur's Inequality, $9q \ge 4p-1$, and so,

$$1-4p+\frac{3q}{p} \ge 1-4p+\frac{3}{p}\cdot\frac{4p-1}{9}=\frac{\left(4p-1\right)\left(1-3p\right)}{3p} \ge 0$$

in the case that $p \in (1/4, 1/3]$.